Semigrupos numéricos

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13 de Janeiro de 2020

Basic notions

- Gaps, non-gaps, genus, gapsets, Frobenius number, conductor
- Generators

Classical problems

- Frobenius' coin problem
- Wilf's conjecture

Counting by genus

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree and Fibonacci bounds
- Ordinarization transform and ordinarization tree
- Quasi-ordinarization transform and quasi-ordinarization forest

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Definition

A numerical semigroup is a subset Λ of $\mathbb{N}_0$ satisfying

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $\#(\mathbb{N}_0 \setminus \Lambda)$ is finite (genus:=g:= $\#(\mathbb{N}_0 \setminus \Lambda)$)

Example: $\{0, 4, 5, 8, 9, 10, 12, \dots\}$

gaps: $\mathbb{N}_0 \setminus \Lambda$
non-gaps: Λ

Definition

[Eliahou-Fromentin] A gapset is a finite subset G of $\mathbb{N}_0$ satisfying

$$\left.\begin{array}{l} a, b \in \mathbb{N}_0 \\ a + b \in G \end{array}\right\} \Longrightarrow a \in G \text{ or } b \in G.$$

G gapset $\Longleftrightarrow$ $\mathbb{N}_0 \setminus G$ numerical semigroup.

Cash point

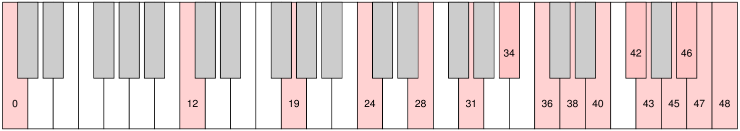
The amounts of money one can obtain from a cash point (divided by 10)

Illustration: Agnès Capella Sala

0 2 4 5 6 7 8 9 10 · · ·

Harmonics: 12-semitone count

Divide the octave into 12 equal semitones.



What semitone interval corresponds to each harmonic?

$$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, 49, 50, \rightarrow\}$$

Definition

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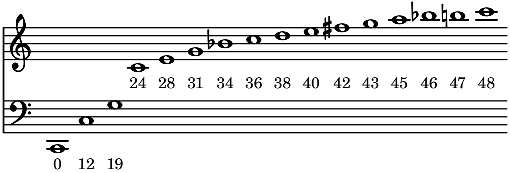
gaps: $\mathbb{N}_0 \setminus \Lambda$
non-gaps: Λ

The third condition implies that there exist

Frobenius number $:=$ the largest gap F
conductor $:= c = F + 1$

The Well-tempered semigroup



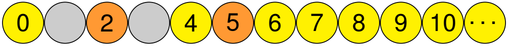
$$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, \dots\}$$

- $g = 33$
- $F = 44$
- $c = 45$

Generators

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps.

Illustration: Agnès Capella Sala



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Frobenius' problem

Suppose we have coins of specified denominations (say $a_1, \ldots, a_n$).

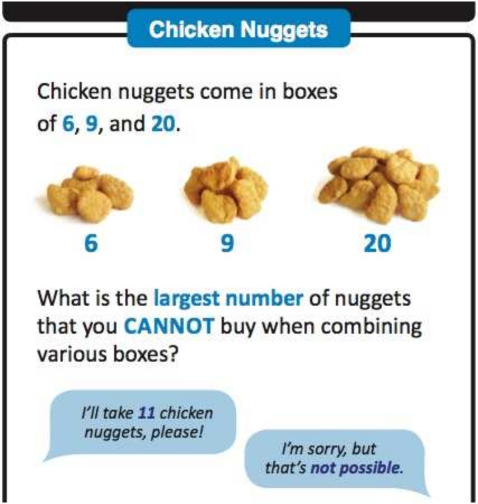
What is the largest monetary amount that can not be obtained using these coins? i.e., $F(\langle a_1, \ldots, a_n \rangle)$?

$n = 2$: Sylvester's formula $a_1 a_2 - a_1 - a_2$.

$n > 2$?

Theorem (Curtis)

There is no polynomial solution for $n = 3$.



i.e. $F(\langle 6, 9, 20 \rangle)$?

Wilf's conjecture (1978)

The number e of generators satisfies

$$e \geqslant \frac{c}{c-g}.$$

Important contributions of Zhai, Eliahou, Dobbs, Matthews, Kaplan, Sammartano, Moscariello, among others.

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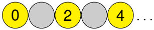
Counting semigroups by genus

Let n_g denote the number of numerical semigroups of genus g.

- $n_0 = 1$, since the unique numerical semigroup of genus 0 is $\mathbb{N}_0$
- $n_1 = 1$, since the unique numerical semigroup of genus 1 is

- $n_2 = 2$. Indeed the unique numerical semigroups of genus 2 are



- $n_3 = 4$
- $n_4 = 7$
- $n_5 = 12$
- $n_6 = 23$
- $n_7 = 39$
- $n_8 = 67$

$\vdots$

Counting semigroups by genus

Conjecture

[B-A 2008]

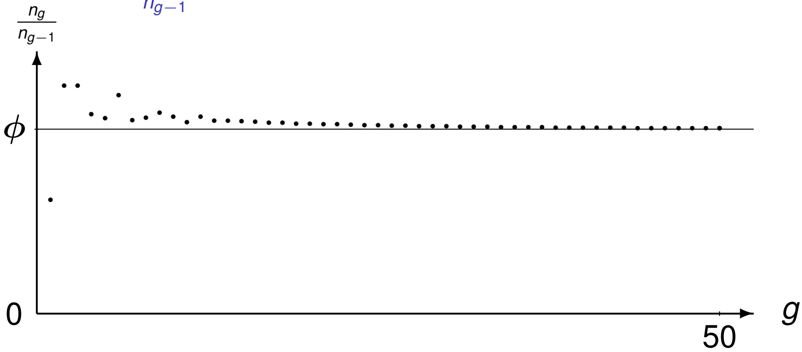
1. $n_g \geqslant n_{g-1} + n_{g-2}$
2. - $\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$
   - $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$

Weaker unsolved conjecture

[B-A 2007] $n_g \leqslant n_{g+1}$

Counting semigroups by genus

Behavior of $\frac{n_g}{n_{g-1}}$



Counting semigroups by genus

What is known

- Upper and lower bounds for n_g

  Dyck paths and Catalan bounds (w. de Mier), semigroup tree and Fibonacci bounds, Elizalde's improvements, and others

- $\lim_{g\to\infty}\frac{n_g}{n_{g-1}}=\phi$

  Alex Zhai (2013) with important contributions of Nathan Kaplan and Yufei Zhao.

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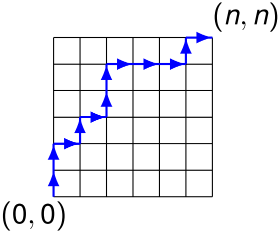
Dyck paths

A Dyck path of order n is a staircase walk from $(0, 0)$ to (n, n) that lies over the diagonal $x = y$.

Example



The number of Dyck paths of order n is given by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

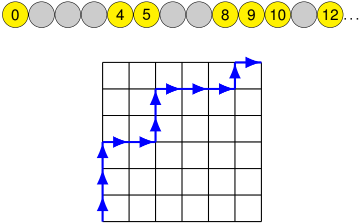
Dyck paths

Definition

The square diagram of a numerical semigroup is the path

$$e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \notin \Lambda, \end{cases} \qquad \text{for } 1 \leqslant i \leqslant 2g.$$

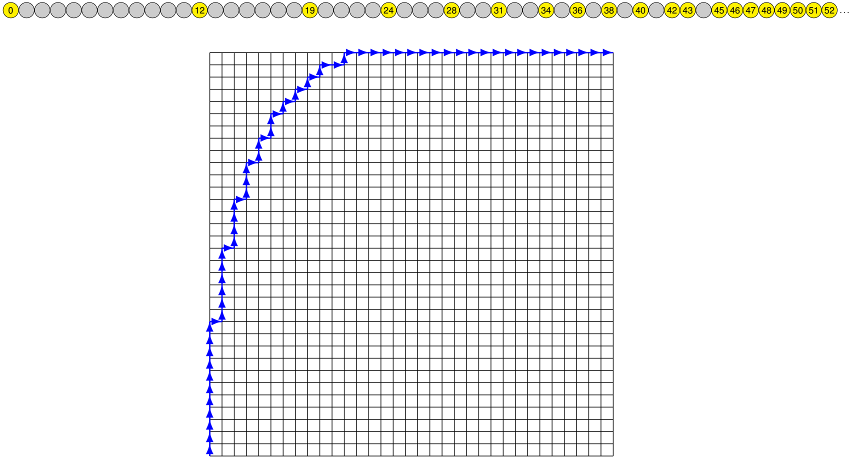
Example



It always goes from $(0, 0)$ to (g, g).

Dyck paths

Example



Dyck paths

Lemma

[B-A, de Mier, 2007]

The square diagram of a numerical semigroup is a Dyck path.

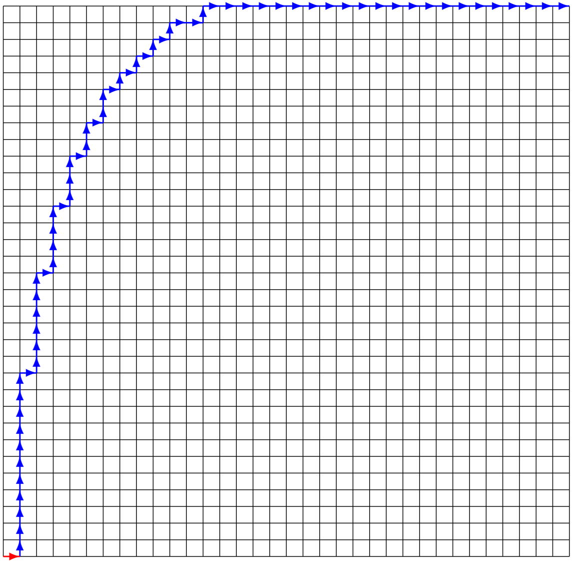
Corollary

$n_g \leqslant C_g = \frac{1}{g+1}\binom{2g}{g}$.

But... not all Dyck paths correspond to numerical semigroups.

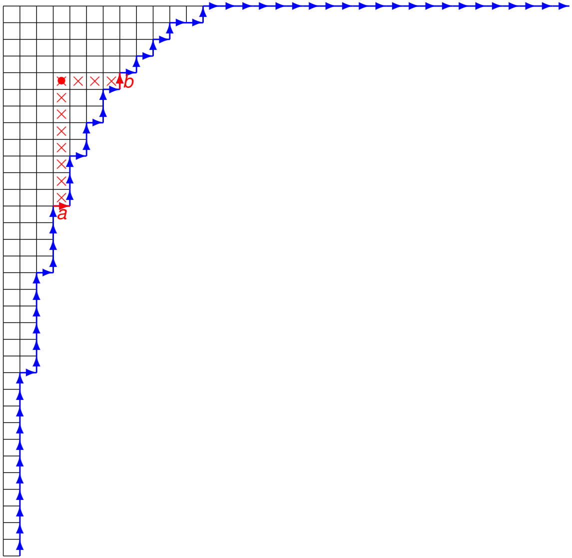
Dyck paths

Use the *augmented Dyck path* (from 0)

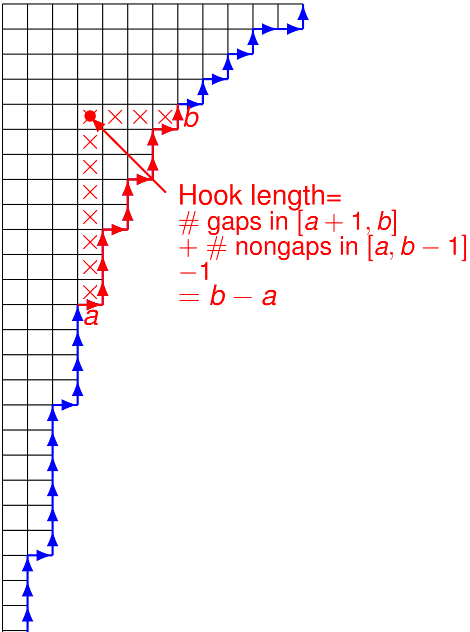


Dyck paths

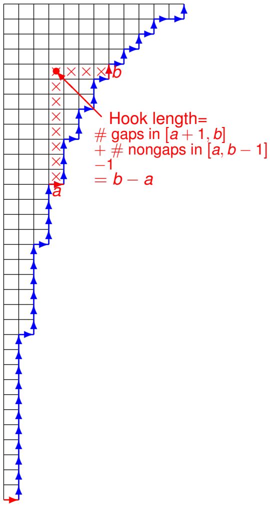
Use the *augmented Dyck path* (from 0) and compute Hook lengths.



Dyck paths



Dyck paths



All Hook lengths

$$H(D) = \{b - a : b \text{ a gap }, a \text{ a nongap}\},$$

Hook lengths of first column

$$h(D) = \{b : b \text{ a gap }\}.$$

By the gapset definition, an (augmented) Dyck path corresponds to a numerical semigroup if and only if

$$H(D) \subseteq h(D)$$

[Constantin, Houston-Edwards, Kaplan]

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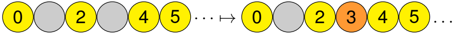
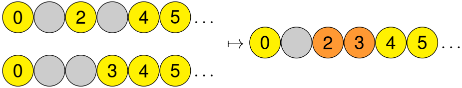
Tree $\mathcal{T}$ of numerical semigroups

From genus g to genus $g-1$

$$\Lambda \mapsto \Lambda \cup \{F(\Lambda)\}$$



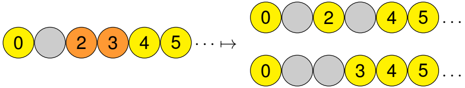
Not injective



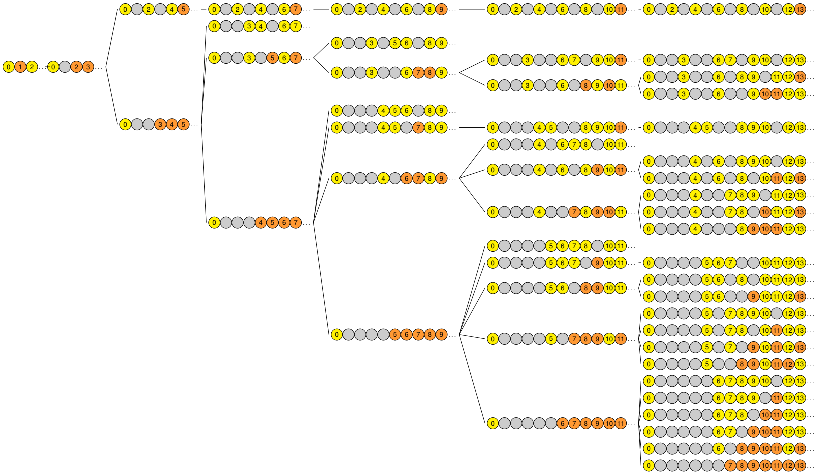
Tree $\mathcal{T}$ of numerical semigroups

From genus $g-1$ to genus g

Take out one by one all generators of Λ larger than $F(\Lambda)$.



Tree $\mathcal{T}$ of numerical semigroups



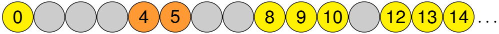
The parent of a semigroup Λ is Λ together with its Frobenius number.

The children of a semigroup are obtained taking away one by one all generators larger than its Frobenius number.

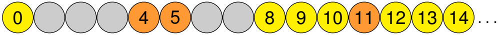
Tree $\mathcal{T}$ of numerical semigroups

If all numerical semigroups had at least one child,
the conjecture $n_g \leqslant n_{g+1}$ would be obvious.

Observe:

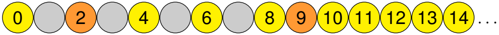


has 0 descendants



has 1 descendants

has 2 descendants



has ∞ descendants

Tree $\mathcal{T}$ of numerical semigroups

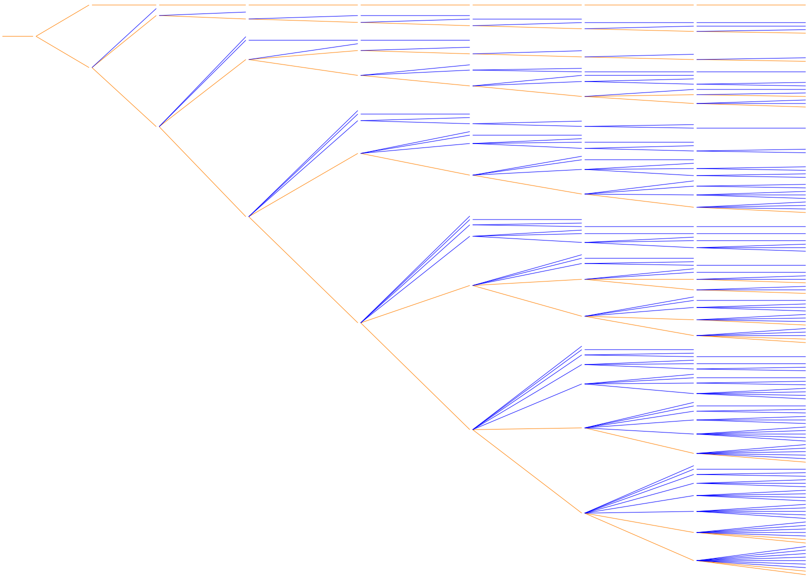
Theorem (B-A, Bulygin, 2009)

Let $d = \gcd(\Lambda \cap [1, c-1])$. Then,

1. *Λ has ∞ descendants $\iff d \neq 1$.*
2. *If $d \neq 1$ then Λ lies in infinitely many infinite chains if and only if d is not prime.*

Computation shows that most numerical semigroups have a finite number of descendants.

Tree $\mathcal{T}$ of numerical semigroups

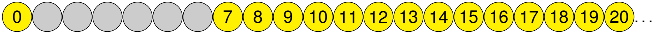


Tree $\mathcal{T}$ of numerical semigroups

We want to analyze the number of children of a node in terms of the number of children of its parent.

Tree $\mathcal{T}$ of numerical semigroups

A numerical semigroup is ordinary if all its gaps are consecutive.



Children of ordinary semigroups

Lemma

If the node of an ordinary semigroup has k children, then its children have $0, 1, \ldots, k-3, k-1, k+1$ children, respectively.

Children of nonordinary semigroups

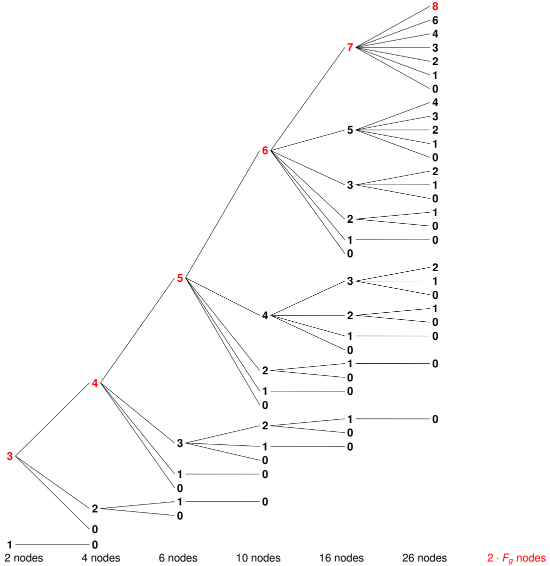
Lemma

If a non-ordinary node in the semigroup tree has k children, then its children have

- *at least $0, \ldots, k-1$ children, respectively,*
- *at most $1, \ldots, k$ children, respectively.*

Subtree

Lower bound for the number of descendants of semigroups of genus $7g$



Subtree

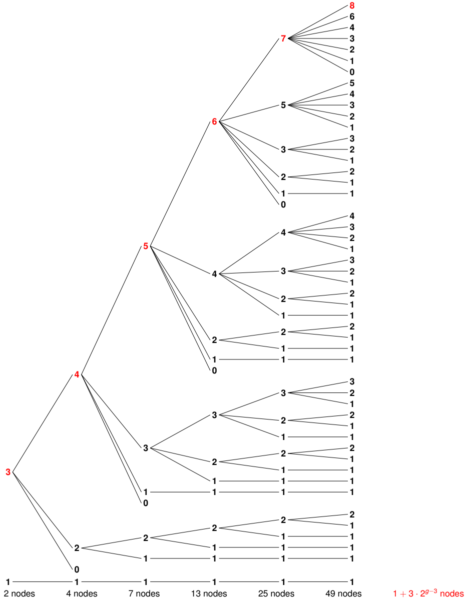
Lemma

For $g \geqslant 3$,

$$2F_g \leqslant n_g$$

Supertree

Upper bound for the number of descendants of semigroups of genus g



Bounds using descending rules

Lemma

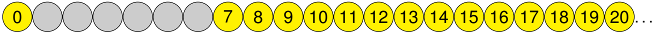
For $g \geqslant 3$,

$$2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}.$$

Ordinary numerical semigroups

The multiplicity of a numerical semigroup is its smallest non-zero nongap.

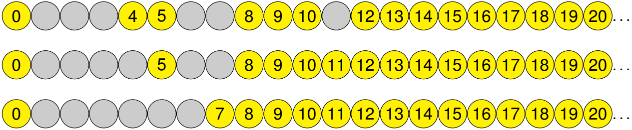
A non-trivial numerical semigroup is ordinary if m=F + 1.



Ordinarization of semigroups

Ordinarization transform of a semigroup:

- Remove the multiplicity
- Add the Frobenius number



- The result is another numerical semigroup.
- The genus is kept constant in all the transforms.
- Repeating several times (:= ordinarization number) we obtain an ordinary semigroup.

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The tree $\mathcal{T}_g$

Define a graph with

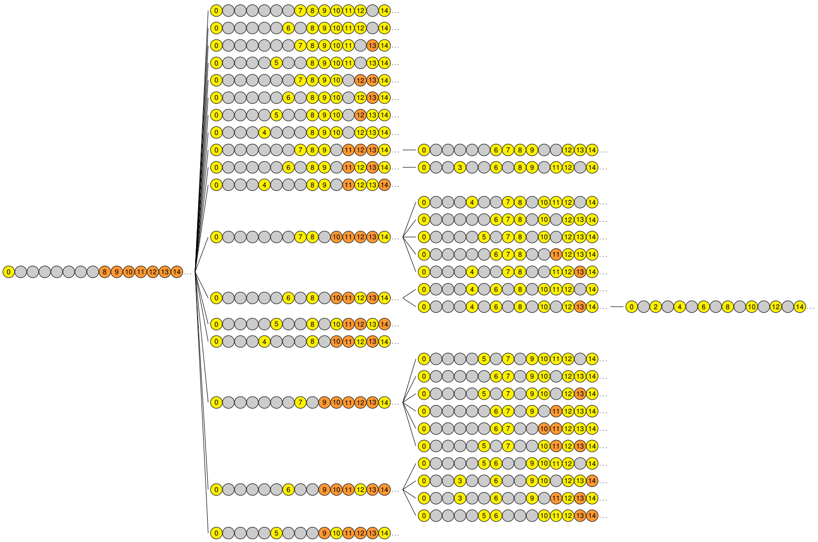
- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its ordinarization transform

$$o(\Lambda) - \Lambda$$

$\mathcal{T}_g$ is a tree rooted at the unique ordinary semigroup of genus g.

Contrary to $\mathcal{T}$, $\mathcal{T}_g$ has only a finite number of nodes (indeed, n_g).

Tree $\mathcal{T}_g$ of numerical semigroups of genus g



Conjecture

$n_{g,r}$: number of semigroups of genus g and ordinarization number r.

Conjecture

- $n_{g,r} \leqslant n_{g+1,r}$
- Equivalently, the number of semigroups in $\mathcal{T}_g$ at a given depth is at most the number of semigroups in $\mathcal{T}_{g+1}$ at the same depth.

This conjecture would prove $n_g \leqslant n_{g+1}$.

This result is proved for the lowest and largest depths.

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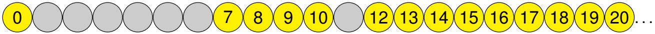
Quasi-ordinary numerical semigroups

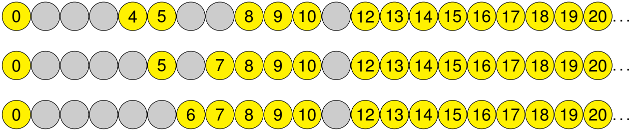
A non-ordinary semigroup Λ is a quasi-ordinary semigroup if $\Lambda \cup F$ is ordinary.



Quasi-ordinarization of semigroups

Quasi-ordinarization transform of a non-ordinary semigroup:

- Remove the multiplicity
- Add the second largest gap



- The result is another numerical semigroup.
- The genus is kept constant in all the transforms.
- Repeating several times (:= quasi-ordinarization number) we obtain a quasi-ordinary semigroup.

Quasi-ordinarization transform of an ordinary semigroup is defined to be itself.

Forest $\mathcal{F}_g$ of numerical semigroups of genus g

The forest $\mathcal{F}_g$

Define a graph with

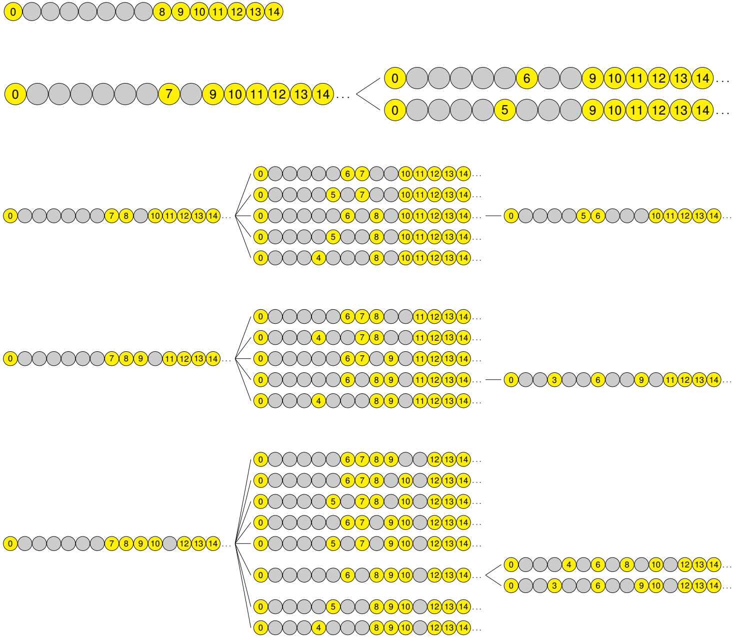
- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its quasi-ordinarization transform

$$q(\Lambda) - \Lambda$$

$\mathcal{F}_g$ is a forest with roots at the quasi-ordinary semigroups of genus g, and the unique ordinary semigroup of genus g.

Contrary to $\mathcal{T}_g$, $\mathcal{F}_g$ is a forest.

Forest $\mathcal{F}_g$ of numerical semigroups of genus g



Conjecture

$n_{g,q}$: $\#$ of semigroups of genus g and quasi-ordinarization number q.

Conjecture

- $n_{g,q} \leqslant n_{g+1,q}$
- Equivalently, the number of semigroups in $\mathcal{F}_g$ at a given depth is at most the number of semigroups in $\mathcal{F}_{g+1}$ at the same depth.

This conjecture would prove $n_g \leqslant n_{g+1}$.

Recommended...

Recommended reference:
Nathan Kaplan. Counting numerical semigroups, Amer. Math. Monthly 124: 862-875, 2017.

Recommended website:
Combinatorial Object Server++ Maintained by Torsten Mütze.

Numerical semigroups arise in

- Algebraic geometry
  (as Weierstrass semigroups, see general references)
- Coding theory
  (see for example Numerical Semigroups and Codes)
- Privacy models
  (see Klara Stokes' PhD thesis and later works)
- Music theory
  (Tempered monoids, the golden fractal monoid, and the well-tempered harmonic se
  (Increasingly Enumerable Submonoids of R: Music Theory as a Unifying Theme,
  to appear in The American Mathematical Monthly)